COMPUTATIONAL ANALYSIS OF ORBITAL MOTION IN GENERAL RELATIVITY AND NEWTONIAN PHYSICS

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1. Overview

This module is intended as a stand-alone component of a second, project-based course in computational science. The students should have had a course in differential equations, and an interest in physics, astronomy or mathematics. It assumes some proficiency with the symbolic, visualization and programming capabilities of Maple, as might be taught in a first course in computational science. The module is implemented in its entirety using Maple.

The learning goals are as follows:

- To review the classical Newtonian theory of orbits.
- To solve, visualize and analyze the Newtonian differential equation whose solutions are Keplerian orbits.
- To modify the Newtonian differential equation to model General Relativity (GR) effects (post-Newtonian correction).
- To introduce the general formalism for GR.
- To see how the modified Newtonian differential equation is consistent with this formalism when we use the exact solution to Einstein's field equations called the *Schwarzschild solution*.
- To apply the formalism to visualize and analyze orbits around a Kerr (rotating) black hole.

2. Introduction to the Problem

The first comprehensive theory of gravitational orbits was developed by Newton. The orbits, or gravitational trajectories, are conic sections and arise as solutions to a second order linear differential equation. Newton assumed that time was absolute and the universe was described by Euclidean geometry. After Newton, Riemann developed the mathematics that describes the geometry of curved spaces [Spivak, 1979]. Einstein adapted this mathematics to describe gravity as the curvature of four-dimensional spacetime [Pais, 1982]. One early success was the application of this theory to explain the advance of the perihelion of Mercury. Soon afterwards, Schwarzschild found the first exact solution to Einstein's field equations. Subsequently this solution and others were interpreted as modeling the gravitational field surrounding a black hole.

3. Statement of the Problem

In this module you will learn how to obtain gravitational trajectories for Newtonian and General Relativistic physics as solutions to differential equations. For Newtonian physics, the relevant differential equation

$$\frac{d^2u}{d\theta^2} + u = p \tag{1}$$

arises from the behavior of a central force. In equation (1), $u = 1/r$ where r is the distance from the central body to the orbiting particle, and p is a constant that can be expressed in terms of the Universal Constant of Gravitation G, the angular momentum of the orbiting particle, and the masses of the central body and the orbiting particle.

For GR, the relevant *system* of differential equations looks like

$$\frac{d^2x^\beta}{d\tau^2} + \sum_{\sigma=1}^{4}\sum_{\alpha=1}^{4} \Gamma^\beta{}_{\sigma\alpha}\frac{dx^\sigma}{d\tau}\frac{dx^\alpha}{d\tau} = 0, \ 1 \le \beta \le 4 \tag{2}$$

and arises from the specification of geodesics, which are a generalization of the notion of the "shortest distance between two points" in a curved geometry. In equation (2), x^1, x^2, x^3, x^4 are spacetime coordinates which depend on the parameter τ and the quantities $\Gamma^\beta{}_{\sigma\alpha}$ are complicated functions of these coordinates.

We note that equation (1) is a single ordinary linear differential equation whereas equation (2) is a system of four ordinary non-linear differential equations.

You will be introduced to the standard notation and formalism for the GR equations and there you will find the definition of $\Gamma^\beta{}_{\sigma\alpha}$ in equation (2). We will also examine solutions to the GR equation (2) specifically for the Schwarzschild and Kerr solutions.

4. Background Information

4.1. Classical Theory of Orbits. The Newtonian gravitational force between two objects of masses m_1 and m_2, separated by a distance r is given by

$$F = G\frac{m_1 m_2}{r^2} \tag{3}$$

where G is the Universal Constant of Gravitation. A straightforward application of Newtonian mechanics starting with this equation yields the differential equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{M}{r^2} \tag{4}$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \tag{5}$$

where $M = G(m_1 + m_2)$. Many problems in classical physics have $m_1 >> m_2$, so we can say that $M \approx Gm_1$. Using equation (5) we have

$$r^2 \frac{d\theta}{dt} = h \tag{6}$$

where h is the angular momentum per unit mass. Combining this with equation (4) we obtain

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{M}{r^2}. \tag{7}$$

It is common in classical orbital mechanics [Fowles and Cassiday, 2005] to make the substitution $u = r^{-1}$. This substitution, together with equation (6), gives us

$$\frac{dr}{dt} = -u^{-2}\frac{du}{dt} = -u^{-2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}.$$

Differentiating, we get

$$\frac{d^2r}{dt^2} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Therefore equation (7) becomes

$$\frac{d^2u}{d\theta^2} + u = p \tag{8}$$

where

$$p = \frac{M}{h^2}. \tag{9}$$

We might think of this differential equation as having the form of a "simple harmonic oscillator." As such the equation is easy to solve and analysis of its solutions is straightforward. We find that the trajectories $r(\theta)$ are conic sections with the central mass M at a focus.

4.2. **A Post Newtonian Correction.** In 1915, Einstein was developing his General Theory of Relativity and was trying to use his theory [Pais, 1982] to explain an anomaly in the orbit of Mercury termed the "perihelion precession." One approach to explaining this anomaly is to search for a modification of equation (8) that somehow reflects corrections to Newtonian gravity. We might be led [Danby, 1988] to the following equation

$$\frac{d^2u}{d\theta^2} + u = p + \epsilon u^2 \tag{10}$$

where ϵ is presumably small and to be determined. It is not difficult to numerically analyze the solutions of this differential equation. We find,

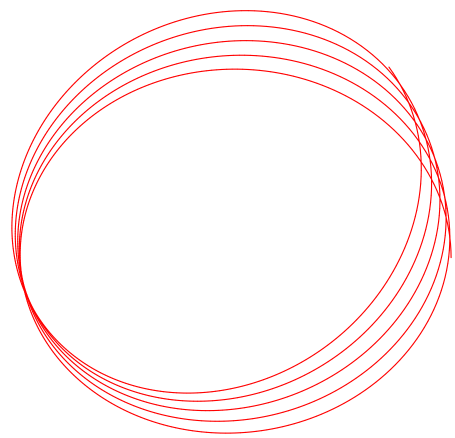

FIGURE 1. A trajectory of equation (10) which corresponds to the famous "advance of the perihelion" of an elliptical orbit.

for example, that for certain values of the parameters, the trajectories resemble an ellipse that "precesses" around. See Figure [1].

In order to symbolically represent the solution to equation (10), we consider the following form:

$$\left(\frac{du}{d\theta}\right)^2 = A + Bu + Cu^2 + Du^3.$$

By differentiating this form, we obtain

$$\frac{d^2u}{d\theta^2} = \frac{B}{2} + Cu + \frac{3D}{2}u^2.$$

Thus we see that equation (10) is equivalent to

$$\left(\frac{du}{d\theta}\right)^2 = A + 2pu - u^2 + \frac{2\epsilon}{3}u^3 \tag{11}$$

where A is related to an initial value of $du/d\theta$, but $3D = 2\epsilon$ is undetermined. This allows us to represent the solution $u(\theta)$ of equation (10) as the inverse function of the function $\theta(u)$ defined by

$$\theta = \int \frac{1}{\sqrt{A + 2pu - u^2 + \frac{2\epsilon u^3}{3}}}\, du. \tag{12}$$

We stress this point: although equation (10) has trajectories that accurately model the anomaly in the orbit of Mercury that Einstein was trying explain in 1915, the assumption of the form of equation (10) as a modification of equation (8) is *ad hoc* and does not give any theoretical explanation of the value of ϵ.

4.3. **Theoretical Context of General Relativity.** General relativity is a refinement of Newton's classical theory of gravitation. In relativity we replace the three-dimensional space continuum of Newtonian physics with a four-dimensional *spacetime* continuum. Instead of "locations" we work with "events" (x, y, z, t). This notational choice suggests Cartesian coordinates and time, but there is nothing special about these coordinates. We may change coordinates, or equivalently, we can expect that different observers will use different coordinates to describe the same abstract collection of events. Thus instead of (x, y, z, t) we are obligated to use a general coordinate notation (x^1, x^2, x^3, x^4).

An important extra feature of spacetime is the presence of what is called a *Lorentz metric*.

At the infinitesimal level, the metric provides a way to ascribe meaning to the "separation" between two infinitesimally separated events. Famous notation for the metric at the infinitesimal level is

$$ds^2 = \sum_{\mu,\nu=1}^{4} g_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

where the $g_{\mu\nu}$ are functions of x^1, x^2, x^3, x^4. Using the so-called *summation convention* of Einstein, we suppress the summation notation and understand from the context of repeated indices that a summation is present. Thus we usually write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

When we can take coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, t)$ for which the metric is of the form

$$ds^2 = -(dx^2 + dy^2 + dz^2) + dt^2,$$

then we are in the case of *special relativity*, and the spacetime is said to be *Minkowski* spacetime.

To understand separation of events in spacetime at a "global" level as opposed to an infinitesimal level, we must consider a path γ in spacetime connecting two events

$$\begin{aligned} \mathbf{x_a} &= (x_a^1, x_a^2, x_a^3, x_a^4) \quad \text{and} \\ \mathbf{x_b} &= (x_b^1, x_b^2, x_b^3, x_b^4). \end{aligned}$$

Thus $\gamma(\tau) = (x^1(\tau), x^2(\tau), x^3(\tau), x^4(\tau))$ for $\tau_0 \le \tau \le \tau_1$ where $\gamma(\tau_0) = \mathbf{x_a}$ and $\gamma(\tau_1) = \mathbf{x_b}$. Then the separation $\Delta s|_{\mathbf{a}}^{\mathbf{b}}$ between these events is

$$\Delta s|_{\mathbf{a}}^{\mathbf{b}} = \int_\gamma ds = \int_{\tau_0}^{\tau_1} \sqrt{g_{\mu\nu}(x^1(\tau), x^2(\tau), x^3(\tau), x^4(\tau))\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau}}\, d\tau.$$

It should be noticed that the separation between two events **a** and **b** depends not only on the two events **a** and **b**, but also on the particular path between the two events.

An important feature of a metric $g_{\mu\nu}$ is its *Riemannian curvature tensor* R_{iklm}. This is a complicated expression involving the derivatives of the functions $g_{\mu\nu}$:

$$\begin{aligned} R_{iklm} &= \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) \\ &\quad + g_{np}\left({\Gamma^n}_{kl}{\Gamma^p}_{im} - {\Gamma^n}_{km}{\Gamma^p}_{il}\right) \end{aligned} \tag{14}$$

where

$$ {\Gamma^a}_{bc} = \frac{1}{2} g^{ai}\left(\frac{\partial g_{ib}}{\partial x^c} + \frac{\partial g_{ic}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^i}\right) \tag{15}$$

and where $(g^{\mu\nu})$ is the inverse of the matrix $(g_{\mu\nu})$. The quantities ${\Gamma^a}_{bc}$ in equation (15) are often called the *Christoffel symbols* or *connection coefficients* of g.

The associated *Ricci curvature tensor* $R_{\mu\nu}$
is given by $R_{\mu\nu} = g^{lm} R_{l\mu m\nu}$, or more explicitly, by

$$R_{\mu\nu} = \frac{\partial {\Gamma^i}_{\mu\nu}}{\partial x^i} - \frac{\partial {\Gamma^i}_{\mu i}}{\partial x^\nu} + {\Gamma^i}_{\mu\nu}{\Gamma^j}_{ij} - {\Gamma^j}_{\mu i}{\Gamma^i}_{\nu j}. \tag{16}$$

For parts of space devoid of matter or energy, Einstein's hypothesis is that the ***Ricci curvature of the metric must be zero***. Close inspection reveals that $R_{\mu\nu} = R_{\nu\mu}$, and thus solving the equations $R_{\mu\nu} = 0$ amounts to the difficult problem of solving ten nonlinear partial differential equations for the ten unknown functions $g_{\mu\nu}$, where $1 \le \mu \le \nu \le 4$.

Einstein hypothesized that once a metric is found whose Ricci curvature is zero, then the trajectories of both particles and light are curves called *geodesics*. Equations for geodesics amount to four ordinary differential equations for the four unknown functions $x^i(\tau)$:

$$\frac{d^2 x^\beta}{d\tau^2} + {\Gamma^\beta}_{\sigma\alpha}\frac{dx^\sigma}{d\tau}\frac{dx^\alpha}{d\tau} = 0. \tag{17}$$

These equations are the same as equation (2) except that we use the summation notion alluded to after equation (13). Notice that for the special case of Minkowski spacetime, the $g_{\mu\nu}$ are constant and thus by equation (15) the quantities ${\Gamma^\beta}_{\sigma\alpha}$ are all equal to zero. Therefore the geodesic equations reduce to

$$\frac{d^2x^\beta}{d\tau^2} = 0,$$

whose solutions correspond to straight lines in (x, y, z, t) Minkowski spacetime.

In equations (17) we choose the parameter τ so that

$$g_{\mu\nu}\left(x^1(\tau_0), x^2(\tau_0), x^3(\tau_0), (x^4(\tau_0)\right) \left.\frac{dx^\mu}{d\tau}\right|_{\tau_0} \left.\frac{dx^\nu}{d\tau}\right|_{\tau_0} = 1. \tag{18}$$

We refer to this choice of parameter as *proper time*. It can be shown [Hughston and Tod, 1990] that if we choose the parametrization of a geodesic so that equation (18) holds at $\tau = \tau_0$, then

$$g_{\mu\nu}\left(x^1(\tau), x^2(\tau), x^3(\tau), (x^4(\tau)\right) \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$$

holds for all values of τ.

In this module we consider only the case where the orbiting particle has non-zero mass. In the case of trajectories for light, we must instead set the righthand side of equation (18) equal to zero.

4.4. **Schwarzschild's Exact Solution to Einstein's Equations.** In 1916, Karl Schwarzschild discovered the first exact solution to Einstein's field equations $R_{\mu\nu} = 0$ for the gravitational field of a point mass. In spherical coordinates $[r\ \phi\ \theta\ t]$ where $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, its metric is given by

$$ds^2 = -\left(\frac{r}{r-2M}\right) dr^2 - r^2 d\phi^2 - r^2 \sin^2\phi\, d\theta^2 + \left(\frac{r-2M}{r}\right) dt^2. \tag{19}$$

See equation (13). For this form of the metric, physical units are chosen so that $G = 1$ and $c = 1$, where G is the universal gravitational constant and where c is the speed of light in the vacuum. In these units, mass will have the units of length. To put this into perspective [Hughston and Tod, 1990], the mass of our sun is about 3km.

Because of the spherical symmetry of the metric, it can be shown that gravitational trajectories are confined to a plane. We assume that the plane is the "equatorial plane" given by $\phi = \pi/2$, and therefore the term $r^2 d\phi^2$ on the right hand side of equation (19) drops out.

In order to find the gravitational trajectories for this solution, we must compute the Christoffel symbols in equation (15) and then solve the geodesic equations (17), where $x^1 = r$, $x^2 = \phi = \pi/2$, $x^3 = \theta$, and $x^4 = t$. It turns out that we can reduce the problem to a single differential equation for $r(\theta)$. Moreover, if we carry out this program, and make the substitution $u = 1/r$ as we did in the Newtonian case in section 4.1, we find that this single differential equation turns out to be

$$\left(\frac{du}{d\theta}\right)^2 = A + 2pu - u^2 + 2Mu^3. \tag{20}$$

We therefore see that the Schwarzschild solution to Einstein's field equations provides a theoretical framework for equations (10) and (11), and tells us how ϵ is related to the mass M. In equation (20), the constant A is usually written as $A = \frac{E^2-1}{h^2}$ where E is the energy of the orbiting particle. In this way we see how the trajectory depends on two physical quantities: energy E and angular momentum h.

4.5. **Kerr's Exact Solution to Einstein's Equations.** Another solution to Einstein's field equations $R_{\mu\nu} = 0$ was discovered in 1963 by Roy Kerr, a New Zealand mathematician. This solution describes the gravitational field of a rotating mass. The Kerr metric, in what are called *Boyer-Lindquist coordinates*, (r, ϕ, θ, t) is

$$\begin{aligned} ds^2 \;&=\; dt^2 - 2Mr\rho^{-2}(dt - a\sin^2\phi\; d\theta)^2 - \rho^2\left(\Delta^{-1}dr^2 + d\phi^2\right) \\ &\quad\; -(r^2 + a^2)\sin^2\phi\; d\theta^2 \end{aligned} \tag{21}$$

In the above equation,

$$\rho^2 = r^2 + a^2\cos^2\phi,\;\; \Delta = r^2 - 2Mr + a^2 \tag{22}$$

The parameter a should be interpreted as the angular momentum per unit mass of the rotating central body of mass M.

The interpretation of the Boyer-Lindquist coordinates is not straightforward. It is certainly true that as $r \to \infty$, the Riemannian curvature of the Kerr metric approaches zero, and so "at infinity" the Boyer-Lindquist coordinates can be thought of as spherical coordinates on the Euclidean space that we get by setting $t =$ constant. Notice that if we let the angular momentum a be zero, this solution collapses to the Schwarzschild solution–see equation (19). In terms of Minkowski spacetime, with coordinates (x, y, z, t), we can think of the condition $r =$ const. as defining a family of nested spheres

$$x^2 + y^2 + z^2 = r^2 \tag{23}$$

that fill up the three-dimensional subspace that we get by setting $t =$ const. So in this sense, where we consider what happens when $a = 0$, we again see that Boyer-Lindquist coordinates act as spherical coordinates for our spacetime.

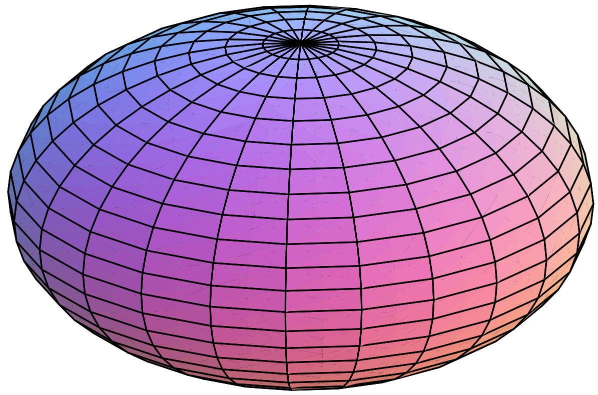

FIGURE 2. An oblate spheroid, which corresponds to a surface of constant r, when $a = 0$ and r is one of the Boyer-Lindquist coordinates.

On the other hand, if we set M to be zero, then we find that the condition $r =$ const. defines a family of nested oblate spheroids

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \tag{24}$$

that fill up the three-dimensional subspace defined by $t =$ const. An "oblate spheroid" is the surface of revolution that we get by revolving an ellipse around its minor axis of symmetry. See Figure 2. To see this, consider the following parameterization of the surface defined by equation (24):

$$\mathbf{R}(\phi, \theta) = \sqrt{r^2 + a^2}\cos\theta\sin\phi\mathbf{i} + \sqrt{r^2 + a^2}\sin\theta\sin\phi\mathbf{j} + r\cos\phi\mathbf{k}. \tag{25}$$

If we represent the metric for this surface as in [Gray, 1998]

$$ds^2 = Ed\phi^2 + 2Fd\phi d\theta + Gd\theta^2$$

then

$$(26)\qquad \begin{aligned} E &= \frac{\partial \mathbf{R}}{\partial \phi}\cdot\frac{\partial \mathbf{R}}{\partial \phi} \\ F &= \frac{\partial \mathbf{R}}{\partial \theta}\cdot\frac{\partial \mathbf{R}}{\partial \phi} \\ G &= \frac{\partial \mathbf{R}}{\partial \theta}\cdot\frac{\partial \mathbf{R}}{\partial \theta}. \end{aligned}$$

A straightforward calculation gives $E = r^2 + a^2\cos^2\phi$, $F = 0$ and $G = (r^2 + a^2)\sin^2\phi$. This is precisely what we get from equation (21) when we put $M = 0$, $dt = 0$ and $dr = 0$. So in this sense, where we consider what happens when $M = 0$, we see that it might be better to think of the Boyer-Lindquist coordinates as acting as oblate-spheroidal coordinates for our spacetime.

It is interesting to note that in equation (24), the condition $r = 0$ corresponds to a disk in the (x, y)-plane of radius a, and the condition $r = 0$, $\theta = \pi/2$ corresponds to the ring

$$(27)\qquad x^2 + y^2 = a^2,\ z = 0$$

In equations (24) and (25), there is no reason to require $r > 0$. In fact when equation (21) is used to study rotating black holes, the Boyer-Lindquist coordinate r is allowed to be negative.

We see that the Kerr metric is not spherically symmetric, it is only axially symmetric. Thus unlike orbital motion for Newtonian physics or for the Schwarzschild solution, *gravitation trajectories for the Kerr solution need not be confined to a plane*. See Figure 3.

For this reason we cannot in general expect to find a description of geodesics that involves only one equation, such as equation (20) for the Schwarzschild solution.

In order to find the gravitational trajectories for the Kerr solution, we may proceed by computing the Christoffel symbols in equation (15) and then solve the geodesic equations (17), where $x^1 = r$, $x^2 = \phi$, $x^3 = \theta$, and $x^4 = t$. It is possible [O'Neill, 1995; Chandrasekhar, 1998] to make a careful analysis of the solutions to equations (17) similar to the derivation of equation (20) from equations (17) for the Schwarzschild solution. For the Kerr solution, the analysis is much more delicate and the results are quite intricate. One key tool used in this analysis [Carter, 1968] is the remarkable *Carter Constant* $\mathcal{K}$. This quantity is a constant of motion for Kerr geodesics, similar to the familiar quantities of energy E and axial component of angular momentum h. See the discussion immediately following equation (20).

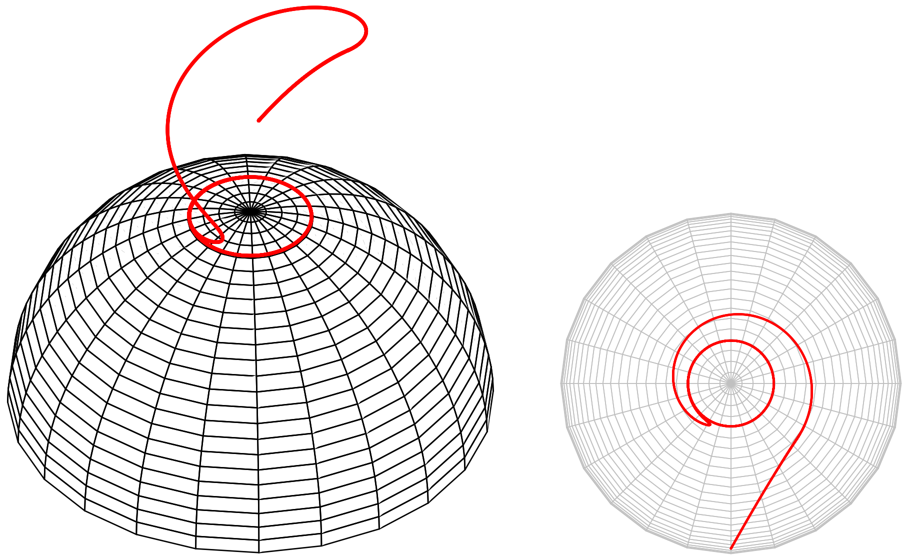

FIGURE 3. A Kerr orbit with $M = 1$, $a = 0.8$, $h = -0.8$, $E = 1$, $\mathcal{K} = 14.4$. The trajectory spirals into the oblate spheroid given by equation (25) with $r = 1.6$, which is the larger of the two roots of Δ. The graph on the right is the view of trajectory when projected down from the z-axis.

For orbits in the Schwarzschild solution, one constant of motion is that the orbit is confined to a plane ($\phi = \text{const}$). This constant of motion not is present in the Kerr solution, but is effectively replaced by the Carter constant. It turns out [Chandrasekhar, 1998] that the geodesic equations (17) can be reduced to the following system of first-order differential equations:

$$\rho^4 \left(\frac{dr}{d\tau}\right)^2 = ((r^2+a^2)E - ha)^2 - \Delta(r^2+\mathcal{K}) \tag{28}$$

$$\rho^4 \left(\frac{d\phi}{d\tau}\right)^2 = \mathcal{K} - a^2\cos^2\phi - (aE\sin\phi - h\csc\phi)^2 \tag{29}$$

$$\rho^2 \frac{d\theta}{d\tau} = \frac{1}{\Delta}\left(2MaEr + \frac{(\rho^2 - 2Mr)h}{\sin^2\phi}\right) \tag{30}$$

$$\rho^2 \frac{dt}{d\tau} = \frac{1}{\Delta}\left(\left((r^2+a^2)^2 - \Delta a^2 \sin^2\phi\right)E - 2aMrh\right) \tag{31}$$

Recall that the quantities $\rho = \rho(r, \phi)$ and $\Delta = \Delta(r)$ are defined in equation (22).

When plotting trajectories using these first-order differential equations, we can no longer specify initial conditions for $r'(\tau)$, $\phi'(\tau)$, $\theta'(\tau)$

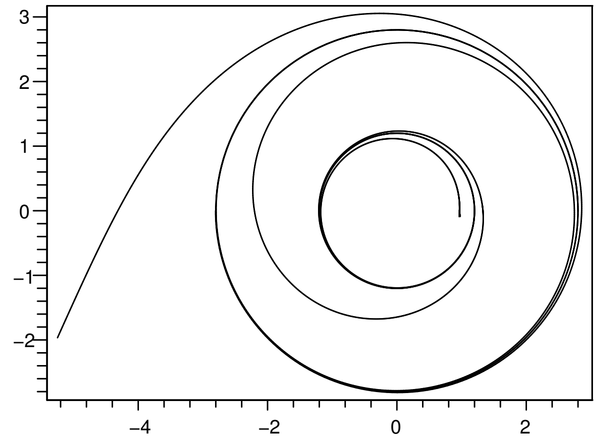

FIGURE 4. An orbit in the Kerr equatorial plane. Notice how the clockwise/counterclockwise sense of the motion changes twice. For this trajectory, $M = 1$, $a = \sqrt{0.84}$, $E = 1.5$, $h = aE$ and the transformation to polar coordinates is based on equation (25) with $\phi = \pi/2$.

and $t'(\tau)$ as we must for the second-order system equations (17). We instead specify values for h, E and $\mathcal{K}$. It should be noted that $r'(\tau)$, $\phi'(\tau)$, $\theta'(\tau)$ and $t'(\tau)$ are related by the condition that

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$$

where $x^1(\tau) = r(\tau)$, etc. See equation (13). This explains why we only need to specify three quantities h, E and $\mathcal{K}$ instead of four. By carefully choosing these parameters, we can plot trajectories using equations (28), (29) and (30).

If we put $\phi(0) = \pi/2$ and $\mathcal{K} = (h - Ea)^2$, then from equation (29) we obtain $d\phi/d\tau = 0$, and so these conditions on h, E, a, K will ensure that the trajectory is confined to the equatorial plane. We can then use equations (28) and (30) to plot trajectories in the equatorial plane. See Figure 4 for an example.

Equations (28) and (30) simplify further if we not only put $\phi(0) = \pi/2$ and $\mathcal{K} = (h - Ea)^2$, but also impose $h = Ea$. We then get

$$\frac{dr}{d\tau} = \pm\frac{\sqrt{E^2 r^2 - \Delta}}{r} \tag{32}$$

$$\frac{d\phi}{d\tau} = 0 \tag{33}$$

$$\frac{d\theta}{d\tau} = \frac{aE}{\Delta} \tag{34}$$

$$\frac{dt}{d\tau} = \frac{E(r^2 + a^2)}{\Delta} \tag{35}$$

5. Conceptual Questions

(1) Do some external reading to discuss contributions of Newton, Riemann, Einstein, Schwarzschild, Kerr, etc. to orbital motion. Two good places to start are [Pais, 1982] and [Spivak, 1979].
(2) In units used in this module, we set the speed of light c and Universal Gravitational constant G equal to 1. Explain why in these units we can measure distance in terms of mass.
(3) Why is equation (8) the appropriate equation for a simple harmonic oscillator?
(4) If we take $M = 0$ in equation (8), then what do we expect to find as the trajectory? Justify your conclusion.
(5) Describe your trajectory in Minkowski spacetime if you are standing still.
(6) The famous apple that fell on Newton's head followed a straight line trajectory. Explain how this trajectory is a special case of equation (8).
(7) What would you expect to see happen to the Schwarzschild solution as $M \to 0$? Make a prediction and carry out a calculation to verify.
(8) A simple model of a curved space is a sphere. What would you expect to find about its curvature? What non-Euclidean geometrical properties would you expect geodesics on this surface to show? What about a cylinder?
(9) What would you expect to see happen to the Kerr solution as $a \to 0$? Make a prediction and carry out a calculation to verify.
(10) What would you expect to see happen to the Kerr solution as $M \to 0$? Make a prediction and carry out a calculation to verify.
(11) With reference to equation (31), explain in words why an observer using Boyer Lindquist coordinates would observe that it would take an infinite amount of time for the particle to reach $r = r_2$, where r_2 is the larger root of $\Delta = 0$.
(12) With reference to equation (30), what happens to the angle θ as the particle approaches $r = r_2$?

6. Problems and Projects

6.1. Symbolic and Numerical Solutions of Newtonian Equations of Motion.

(1) Use Maple to find an exact solution to the differential equation (8).
   (a) Take $p = 1$, and plot some trajectories in the plane corresponding to various initial conditions $u0 = u(0)$ and $Du0 = u'(0)$. Try to find initial conditions that result in ellipses and hyperbolae.
   (b) Depending upon the values of $u0$, $Du0$ and p, the trajectory will be an ellipse, parabola, or hyperbola. Describe the points in the $(u0, Du0)$-plane for which the trajectory is an ellipse. You may assume $u0 > 0$.

(2) Next use Maple to find a numerical representation of the solution to the differential equation (8). Using this representation, plot elliptic, parabolic and hyperbolic trajectories.

6.2. Numerical Solutions of Post-Newtonian Equations of Motion.

(1) Use Maple to find a numerical representation of the solution to the differential equation (10).

   Use this representation to plot trajectories for the parameter values shown in Figure (5).

p	ϵ	$u(0)$	$u'(0)$
1	0	1	0.7
1	0.005	1	0.7
1	0.01	1	0.7
1	0.02	1	1
1	0.3	1	0.1

Figure 5. Parameter values for trajectories in Section 6.2

(2) Go through the steps necessary to verify that equation (10) is equivalent to equation (11).

(3) The classification of the trajectories for equation (10) can be approached by careful analysis [Chandrasekhar, 1998] of the roots of the cubic polynomial on the right hand side of equation (11). If we specify p, ϵ, $u(0)$ and $u'(0)$, then we can determine A.

   Plot the resulting cubic function of u for values of the parameters in Figure (5). Choose a scale that clearly shows the positive roots. Indicate which part of the cubic curve corresponds to the trajectory.

6.3. **Maple Representation of Metric Properties.**

6.3.1. *Maple Representation of a Spacetime Metric.* Use Maple to represent general coordinates $X = [x_1, x_2, x_3, x_4]$, with a Maple Array, and set up $g_{\mu\nu}$ as a symmetric matrix whose entries are arbitrary expressions of x_1, x_2, x_3, x_4. Treat g as a Maple Matrix.

Illustrate your work by using it to set up a "perturbation" of the Minkowski metric where $g_{\mu\nu}(x_1, x_2, x_3, x_4)$ is given by the matrix

$$\begin{bmatrix} -1+\epsilon\, x_1 & \epsilon\,(x_1+x_2) & 0 & 0 \\ \epsilon\,(x_1+x_2) & -1+\epsilon\, x_2 & 0 & 0 \\ 0 & 0 & -1+\epsilon\, x_3 & 0 \\ 0 & 0 & 0 & 1+\epsilon\, x_4 \end{bmatrix}. \tag{36}$$

6.3.2. *Computation of Christoffel Symbols.* Write a Maple worksheet that takes general expressions $g_{\mu\nu}(x^1, x^2, x^3, x^4)$, and computes the quantities ${\Gamma^a}_{bc}$. See equation (15). Treat these quantities as a Maple Array. Each entry of your Array should be a *procedure* representing a function of (x_1, x_2, x_3, x_4). Use Maple to print the non-zero components of your array when g is the perturbed Minkowski metric (36).

6.3.3. *Computation of the Riemannian Curvature.* Write a Maple worksheet that takes general expressions $g_{\mu\nu}(x^1, x^2, x^3, x^4)$, and computes the quantities R_{iklm} of the Riemannian curvature tensor as well as the quantities $R_{\mu\nu}$ of the Ricci curvature tensor. See equations (14) and (16). Treat Riemannian and Ricci curvature as Maple Arrays. Each entry of an Array should be a *procedure* representing a function of (x_1, x_2, x_3, x_4). Use Maple print the non-zero components of your arrays when g is the perturbed Minkowski metric (36).

6.4. **The Schwarzschild Solution.**

6.4.1. *Metric Properties.* Use your solutions to previous projects to verify that the Schwarzschild metric has vanishing Ricci curvature. The Schwarzschild metric can be represented in spherical coordinates $[r\ \phi\ \theta\ t]$ $(0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi)$ as

$$g = \begin{bmatrix} -\frac{r}{r-2\,M} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2(\phi) & 0 \\ 0 & 0 & 0 & \frac{r-2\,M}{r} \end{bmatrix}. \tag{37}$$

See equation (19). Also, print the nonzero Christoffel symbols and nonzero components of the Riemannian curvature.

6.4.2. *Schwarzschild Trajectories.*

(1) Set up the system of equations (17) in Maple and use **DEplot** to plot the trajectory corresponding to the middle row of Figure (5). The solution is a parametric curve $(r(\tau), \phi(\tau), \theta(\tau), t(\tau))$ in spacetime, and to plot the curve $[r(\tau), \theta(\tau)]$ you can use **DEplot** with the option **scene**, together with **transform** to convert the trajectory to a polar curve.
Assume $\phi(0) = 0$ and $\phi'(0) = 0$. This will confine the motion to the equatorial plane.
You will need to work out M and the initial conditions

$$[R(0), R'(0), \theta(0), \theta'(0), t(0), t'(0)].$$

To do this, you will need to use the fact that $M = 2\epsilon$, $R = 1/u$, and equations (6), (9). Note: modify equation (6) to be

$$r^2 \frac{d\theta}{d\tau} = h. \tag{38}$$

You will also need to use the following form of equation (18):

$$g_{11}r'(0)^2 + g_{22}\phi'(0)^2 + g_{33}\theta'(0)^2 + g_{44}t'(0)^2 = 1.$$

(2) Derive equation (20) from the system of equations (17). This validates the post-Newtonian correction equation (10) and its equivalent form equation (11). You may find the following outline useful.
   (a) Generate the four geodesic equations with **seq**, and represent them as a list **diffeqs**.
   (b) You should note that one of them is

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} - \sin(\phi)\cos(\phi)\left(\frac{d\theta}{d\tau}\right)^2 = 0$$

   Notice that if $\phi(\tau) = \pi/2$, then this differential equation is trivially satisfied. Physically this corresponds to the fact that if a trajectory begins in the equatorial plane, then it remains in this plane.
   This suggests setting $\phi(\tau) = \pi/2$ with the syntax
   >phi:=tau→ Pi/2;
   Do so, and when you look at **diffeqs** you should see that one of the equations drops out.

(c) Next notice that two of three remaining differential equations (the ones involving $\frac{d^2t}{d\tau^2}$ and $\frac{d^2\theta}{d\tau^2}$) can be reduced to first order equations for $\frac{dt}{d\tau}$ and $\frac{d\theta}{d\tau}$ by a substitution. Make the appropriate substitution in each equation, and solve the two differential equations. This will give you $\frac{dt}{d\tau}$ and $\frac{d\theta}{d\tau}$ in terms of $r(\tau)$.

(d) Substitute these expressions into **diffeqs** for $\frac{dt}{d\tau}$ and $\frac{d\theta}{d\tau}$. You should now find that **diffeqs** reduces to one differential equation for $r(\tau)$.

(e) Use Maple to solve this differential equation. From the resulting representation for the solution, you can easily see how to get a formula for $\left(\frac{d\tau}{dr}\right)^2$.

(f) Using this expression for $\left(\frac{d\tau}{dr}\right)^2$ in terms of $r(\tau)$ and the expression from item (2c) above for $\left(\frac{d\theta}{d\tau}\right)^2$, you can get an expression for $\left(\frac{dr}{d\theta}\right)^2$.

(g) Finally make the substitution $u(\theta) = 1/r(\theta)$ into the resulting expression for $\left(\frac{dr}{d\theta}\right)^2$ and clean up the result.

6.5. **The Kerr Solution.**

6.5.1. *Metric Properties.* Just as you did for the Schwarzschild metric, verify that the Kerr metric has vanishing Ricci curvature. The matrix representation of the Kerr metric in Boyer-Lindquist coordinates $[r\ \phi\ \theta\ t]$ is

(39)

$$\begin{bmatrix} -\dfrac{\rho^2}{\Delta} & 0 & 0 & 0 \\ 0 & -\rho^2 & 0 & 0 \\ 0 & 0 & -\left(r^2 + a^2 + \dfrac{2Mra^2\sin^2\phi}{\rho^2}\right)\sin^2\phi & \dfrac{2Mra\sin^2\phi}{\rho^2} \\ 0 & 0 & \dfrac{2Mra\sin^2\phi}{\rho^2} & 1 - \dfrac{2Mr}{\rho^2} \end{bmatrix}$$

6.5.2. *Kerr Trajectories.*

(1) Set up the system of second-order geodesic differential equations (17) in Maple and use **DEplot** to plot the trajectory corresponding to the following initial conditions:

$$\begin{aligned} r(0) &= 2, & r'(0) &= -0.5291502622 \\ \phi(0) &= \pi/2, & \phi'(0) &= 0 \\ \theta(0) &= -\pi/2, & \theta'(0) &= -0.7637626155 \\ t(0) &= 0, & t'(0) &= -4.033333334 \end{aligned}$$

The four derivatives satisfy equation (18).
For this problem take $M = 1$ and $a = \sqrt{0.84}$.

The solution is a parametric curve $(r(\tau), \phi(\tau), \theta(\tau), t(\tau))$ in spacetime that is confined to the plane $\phi = \pi/2$ and spirals into a circle whose radius is the larger root of $\Delta(r) = 0$. See equation (22).

To plot the curve $[r(\tau), \theta(\tau), \phi(\tau)]$ use **DEplot3d** with the option **scene**, together with **transform** to convert the trajectory to a polar curve. Take τ to be between 0 and 0.982.

Now plot the remaining inner trajectories. Use the initial conditions provided below:

- For the trajectory between the two roots of $\Delta(r)$, namely $r = 0.6$ and $r = 1.4$, use the following initial conditions:

$$\begin{aligned} r(0) &= 1.39, & r'(0) &= -0.702914514 \\ \phi(0) &= \pi/2, & \phi'(0) &= 0 \\ \theta(0) &= -\pi/2, & \theta'(0) &= 81.21020225 \\ t(0) &= 0, & t'(0) &= 245.6290861 \end{aligned}$$

- Now, for the inner most trajectory, use the following initial conditions:

$$\begin{aligned} r(0) &= 0.59, & r'(0) &= -0.683177008 \\ \phi(0) &= \pi/2, & \phi'(0) &= 0 \\ \theta(0) &= -\pi/2, & \theta'(0) &= -79.20501206 \\ t(0) &= 0, & t'(0) &= -102.6752865 \end{aligned}$$

(2) Set up the system of first-order differential equations (28)–(30) as functions of $h, E, \mathcal{K}$ and plot the orbit corresponding to $h = aE$, $E = 0.7$ and $\mathcal{K} = 0$. If you take $\phi_0 = \pi/2$, then these conditions guarantee an orbit confined to the equatorial plane $\phi = \pi/2$.

You will need to do this in three plots. If r_1 and r_2 are the roots of Δ with $r_1 < r_2$, then you will need one plot for $r > r_2$, another for $r_1 < r < r_2$ and yet another for $r < r_1$. Set $M = 1$ and $a = \sqrt{0.84}$. To get started, take $r(0) = 1.6$.

(3) The initial conditions on $r'(0)$, $\phi'(0)$ and $\theta'(0)$ in item (1) above are synchronized with the values for h, E and $\mathcal{K}$ in item (2) above. Verify this.

This partially validates the equivalence of the second-order geodesic equations used in item (1) and the first-order system in item (2).

(4) Use your setup of the system of equations (28)–(30) as functions of $h, E, \mathcal{K}$ and plot Kerr orbits that are not confined to a plane. You can begin with the parameters given in Figure 3. In that figure, $r(0) = 4.0$, $\phi(0) = \pi/8$, $\theta(0) = 0$ and the range for τ is from -4.5157703 to 0.

Glossary: Computational Analysis of Orbital Motion

Boyer-Lindquist coordinates: Coordinates frequently used to describe the Kerr solution. At spatial infinity in the Kerr spacetime, they behave like spherical coordinates for Euclidean space. They were introduced about four years after Kerr's initial discovery of the Kerr metric.

Christoffel symbols: When a spacetime has curvature, these quantities allow us to define differentiation in a way that is independent of the choice of coordinates. The Christoffel symbols can be expressed in terms of the derivatives of the metric components, and all of the important mathematical quantities of a spacetime such as curvature and geodesic equations can be expressed in terms of the Christoffel symbols. The Christoffel symbols are a three-index system of quantities, but it is important to note that they do not form a tensor.

constant of motion: When a particle moves along an orbit certain physical quantities are often conserved. In Newton's classical theory of orbits, the most basic example is that a Newtonian orbit (for central forces) is confined to a plane: the spherical coordinate variable ϕ is constant. Another example is the angular momentum h of the orbiting body. The energy of the particle is also constant: as the orbiting body moves farther away from the central force it slows down and so its kinetic energy decreases, but this is offset by a gain in potential energy. When a second-order system such as equations (17) can be reduced to a first-order system such as equations (29–31), typically the initial conditions on the derivatives get replaced by constants of motion.

Einstein's field equations: The system of partial differential equations in General Relativity that equate the Ricci curvature of the metric with the stress energy tensor. For empty space, the stress energy tensor vanishes and the equations amount to equating the Ricci curvature of the metric to zero. The field equations are a system of ten coupled, nonlinear partial differential equations.

geodesic: A generalization of the notion of the shortest distance between two points in a curved space. Specifically, it is an extremal point of the arc-length function between two points. In

General Relativity geodesics maximize the separation between two events.

Keplerian orbit: A classical gravitational trajectory that is a conic section with one focus at the central mass. Kepler formulated three empirical laws that describe the orbital motion of planets. Later Newton used his theory of gravitation to give a theoretical explanation for Kepler's Laws.

Kerr Solution: A metric describing the general relativistic gravitational field of a rotating mass. The solution is axial symmetric and its Ricci curvature vanishes. It was discovered by Roy Kerr in 1963.

Lorentz metric: The metric that gives the separation between two events in Minkowski space. It is given by $ds^2 = -(dx^2 + dy^2 + dz^2) + dt^2$ in units where the speed of light is equal to one. The Riemannian curvature of the Lorentz metric is zero.

metric: A measure of the infinitesimal separation between two points in a curved space. It is a generalization of the Euclidean distance formula.

Minkowski spacetime: The spacetime that is the arena for special relativity. Minkowski spacetime admits global coordinates (x, y, z, t) where (x, y, z) is a spatial location at t is a moment in time. An object (x, y, z, t) is called an *event*. The Lorentz metric for Minkowski spacetime is given by $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, where c is the speed of light. This metric has vanishing Riemannian curvature. Minkowski spacetime is the simplest spacetime that models a universe with no gravitation. A general curved spacetime has the property that the tangent space at each point is a Minkowski spacetime.

Oblate spheroidal coordinates: A system of coordinates for three-dimensional Euclidean space that generalizes spherical coordinates. The round spheres become oblate spheroids, and the cones $\phi = c$ of spherical coordinates become hyperboloids of revolution. This coordinate system is often used in geology and atmospheric physics since the earth bulges at the equator, and is thus often modeled as an oblate spheroid.

perihelion: In Newtonian mechanics, the minimal orbital distance of a planet from the sun.

Post Newtonian Correction: A modification of Newtonian physics that attempts to account for non-Newtonian anomalies as perturbations to Newtonian theory.

proper time: Time between two events, measured by a clock, moving with an observer traveling from one event to the other through spacetime.

Ricci curvature: A trace of the Riemannian curvature. The field equations in a vacuum require that it vanish there. It is convenient to think about it as a 4×4 matrix.

Riemannian curvature: The mathematical object that quantifies the curvature of a spacetime. It can be expressed in terms of the derivatives of the coefficients of the metric. It is often thought of as a four-index tensor. Although it has 256 components, there are many symmetries amongst the indices, and it turns out there are only 20 independent components. One can think of the Ricci curvature as a trace of the Riemannian curvature. Einstein's field equations require that the Ricci curvature of a spacetime to be zero, but this in no way forces the Riemannian curvature to be zero.

Riemannian geometry: The mathematics developed by Riemann that describes geometry in an n-dimensional curved space. Riemann was a student of Gauss, who laid the foundations for Riemannian geometry by his work on the curvature of surfaces in space. In Riemannian geometry, ds^2, which measures the separation between two infinitesimally separated points in the curved space, is typically a positive definite quadratic form. When the formalism of Riemannian geometry is applied to General Relativity, $n = 4$ and the quadratic form is no longer positive definite, but is indefinite with signature $(1, 3)$.

Schwarzschild solution: A metric describing the general relativistic gravitational field of a point mass. The solution is spherically symmetric and its Ricci curvature vanishes. It was the first non-trivial exact solution to the field equations to be discovered.

spacetime: A unification of the three spatial dimensions with time. Points in spacetime are called *events*. Spacetime is endowed with a Lorentz metric that allows one to compute the *separation* between events. This notion of separation between events replaces the both the notion of distance between spatial locations and the time between events. A spacetime might have nonzero curvature. Minkowski spacetime, the arena for special relativity, has vanishing curvature. A spacetime whose Ricci curvature equals zero is a spacetime that satisfies the Einstein field equations.

special relativity: A theory that analyzes the separations between events in Minkowski spacetime. This theory was originally developed by Albert Einstein in 1905. It provides a model of time dilation and length contraction at high relative speeds in non-accelerating and non-gravitational reference frames.

spherical coordinates: A system of coordinates for Euclidean space appropriate for situations involving spherical symmetry. In this module, θ is the longitudinal angle that has a range of 2π, and ϕ is the co-latitudinal angle that has a range of π. The equatorial plane is given by $\phi = \pi/2$, and the positive z-axis is given by $\phi = 0$. In many textbooks and articles, especially those written by physicists, the roles of θ and ϕ are reversed. Care must be taken when reading the literature to understand which convention is being used.

summation convention: In general relativity, physical quantities are often arrays that are denoted with *index* notation, such as R_{ij}, R^i_{jkl}, g_{ij}, or g^{ij}. If two such quantities are juxtaposed with repeated indices, then it is understood that the repeated indices are summed, especially if one of the matching indices is *up* and the other is *down*. For example, $A^{ij}B_{jk}$ is the quantity $C^i_k = \sum_j A^{ij}B_{jk}$. WARNING: $D_{ik} + A^{ij}B_{jk}$ means $D_{ik} + \sum_j A^{ij}B_{jk}$ and not $\sum_j D_{ik} + A^{ij}B_{jk}$.

Universal Constant of Gravitation: The scaling factor for Newtonian gravitational attraction. Its current experimental value (as of 2002) is 6.6742×10^{-11} m^3 kg^{-1} s^{-2}

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